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ABSTRACT: The motion of two spheres in an ideal fluid is studied. The kinetic energy and the hydrodynamic interaction forces are calculated for the case of small distance between the spheres, in particular for the case of contacting spheres. The velocity field for contacting spheres is determined.

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ON THE MOTION OF TWO SPHERES IN AN IDEAL FLUID

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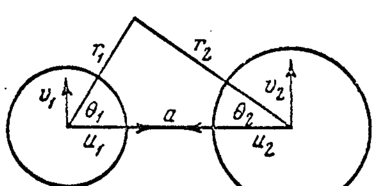
A study is made of the motion of two spheres in an ideal fluid. The kinetic energy and the forces of hydrodynamic interaction are calculated for the case when the distance between spheres is small, in particular for contact between the spheres. The features of the velocity field upon contact between the spheres are determined. /659*

The kinetic energy of the fluid for the case of sphere motion along a line connecting centers (center line) was calculated by Hicks [1]. Upon the motion of spheres perpendicular to the center line the kinetic energy is known when the distance between spheres is considerably greater than their radii [2].

1. Velocity Potential. Two spheres are moving in an ideal incompressible fluid at rest at infinity. The motion of the fluid is assumed to be potential. In calculating the velocity potential it is sufficient, by virtue of the linearity of the problem, to consider the case when the velocities of the spheres are coplanar.

Spherical coordinate systems r_i, θ_i, φ_i are chosen with origin at the center of the i-th sphere (i = 1, 2) and with positive direction of the polar axes toward the adjacent sphere (see the Figure). The azimuth angle φ_i is measured from the direction perpendicular to the velocities of the spheres. The positive direction of the polar axis of the i-th coordinate system is chosen as the positive direction of projection u_i of the velocity onto the line joining centers. The positive directions of the projections v_1 and v_2 of the velocities of the spheres onto a line perpendicular to the line joining centers are chosen to coincide.



The velocity potential Φ of the fluid must staisfy the Laplace equation in a region exterior to the two spheres and the boundary conditions

$$\Delta\Phi = 0, \quad \partial\Phi/\partial r_i|_{r_i=R_i} = u_i \cos\theta_i + v_i \sin\theta_i \sin\varphi_i$$
$$\Phi \to 0 \quad \text{as} \quad r_i \to \infty$$

*Numbers in the margin indicate pagination in the foreign text.

where R_i is the sphere radius. The solution of this problem can be found by the method of images [1-4]. The potential is determined by successive approximation and is the sum of a series of functions Φ_n^i harmonic in the exterior of the i-th sphere

$$\Phi = (\Phi_0^1 + \Phi_1^1 + \Phi_2^1 + \ldots) + (\Phi_0^2 + \Phi_1^2 + \Phi_2^2 + \ldots) \tag{1.1}$$

In this case Φ_n^i satisfies the following conditions on the i-th sphere: /660

$$\partial \Phi_0^i / \partial r_i = u_i \cos\theta_i + v_i \sin\theta_i \sin\varphi_i \tag{1.2}$$

$$\begin{gathered} \partial \Phi_n^i / \partial r_i = -\partial \Phi_{n-1}^k / \partial r_i \quad (n = 1, 2, \ldots) \\ \Phi_n^i \to 0 \quad \text{as} \quad r_i \to \infty \quad (n = 0, 1, \ldots) \end{gathered} \tag{1.3}$$

Here and everywhere below $k = 1, 2, k \neq i$.

First of all it is possible to examine the motion of spheres along a line joining centers ($v_1 = v_2 = 0$). In this case it is well known [1-4] that the functions Φ_n^i will be the potentials of dipoles located within the spheres along the center line. It is possible to seek the Φ_n^i in the following form:

$$\Phi_n^i = \alpha_n^i (r_i \cos\theta_i - a_{in})(r_i^2 - 2r_i a_{in} \cos\theta_i + a_{in}^2)^{-3/2} \tag{1.4}$$

Substitution of (1.4) into (1.3) makes it possible to find equations for the unknown coordinates and for the strength of the dipoles:

$$a_{in}(a - a_{kn-1}) = R_i^2, \quad a_{i0} = 0 \tag{1.5}$$

$$\alpha_n^i = \alpha_{n-1}^k (a_{in} / R_i)^3, \quad 2\alpha_0^i = -u_i R_i^3 \tag{1.6}$$

Here a is the distance between the centers of the spheres. These recurrence relations can be solved most simply if we seek not the coordinates a_{in}, but the products of coordinates, in the same way as did Murphy in the electrostatic problem of the potential of two charged spheres [5]. Introduced here are the new coefficients A_n^i and B_n^i, defined by the formulas

$$2\alpha_{2n}^i = -u_i (R_i / A_n^i)^3, \qquad 2\alpha_{2n-1}^i = -u_k (R_k / B_n^i)^3 \tag{1.7}$$

Then, according to (1.6) the coordinates of the dipoles are

$$a_{i2n} = R_i B_n^k / A_n^i, \qquad a_{i2n-1} = R_i A_{n-1}^k / B_n^i \tag{1.8}$$

The coefficients A_n^i and B_n^i are determined from (1.5)-(1.8) in the following form:

$$(\tau - \tau^{-1})A_n^i = \tau^n (\tau + R_i / R_k) - \tau^{-n}(\tau^{-1} + R_i / R_k),$$
$$(\tau - \tau^{-1})B_n^i = (\tau^n - \tau^{-n})\, a / R_i \qquad (1.9)$$

Here τ is the root of the equation

$$a^2\tau = (\tau R_1 + R_2)(\tau R_2 + R_1) \qquad (1.10)$$

Actually, substitution of (1.8) into (1.5) yields the recurrence relations

$$R_i B_n^i + R_k B_{n-1}^i = aA_{n-1}^k, \quad R_i A_n^i + R_k A_{n-1}^i = aB_n^k \qquad (1.11)$$

with initial conditions

$$A_0^i = 1, \quad A_1^i = (a^2 - R_k^2)/R_1R_2; \quad B_0^i = 0, \quad B_1^i = a/R_i \qquad (1.12)$$

Relations (1.11) are solved for A_n^i and B_n^i. In particular, for A_n^i we get

$$A_n^i - A_{n-1}^i (a^2 - R_1^2 - R_2^2) / R_1R_2 + A_{n-2}^i = 0 \qquad (1.13)$$

The coefficient B_n^i satisfies exactly the same equation. The general solution of /661
the recurrent chain (1.13) with arbitrary conditions in the two numbers is $c_1\tau^n + c_2\tau^{-n}$, where τ is determined by (1.10). The constants c_1 and c_2 are determined from the initial conditions (1.12), and as a result we get formulas (1.9).

The convergence of series (1.1) with functions (1.4) takes place everywhere, except for the point of tangency of the spheres $\theta_i = 0$, $r_i = R_i$, when the spheres are in contact. In this case the condensation point of the coordinates of the dipoles a_{in} changes to the point of tangency of the spheres. This factor is the reason for the ineffectiveness of the method of expansion in spherical functions used in [2], when the distance between spheres is relatively small compared to the radius. In the region with eliminated point of tangency the potential series converges approximately as $1/n^3$. If the spheres do not touch each other, then, as follows from (1.5)-(1.6), the series converges approximately as a geometric progression whose index decreases rapidly with decreasing distance between spheres.

2. Tangential Velocity on Spheres in Contact. On spheres the formulas for the potential, (1.1) and (1.4), are simplified appreciably if account is taken of (1.5) and (1.6):

$$\Phi|_{R_i} = \frac{\alpha_0^i \cos\theta_i}{R_i^2} + \sum_{n=1}^{\infty} \alpha_n^i \left(\frac{R_i^2}{a_{in}} - a_{in}\right)(R_i^2 - 2R_i a_{in}\cos\theta_i + a_{in}^2)^{-3/2} \qquad (2.1)$$

If the spheres are in contact, $\tau = 1$ follows from formula (1.10). In this case we get from (1.9)

$$B_n^i = na / R_i, \quad A_n^i = 1 + na / R_k \tag{2.2}$$

Substitution of (2.2) into (1.8) makes it possible to find the coordinates of the dipoles

$$a_{i2n} = R_i / (1 + R_k / an), \quad a_{i2n-1} = R_i (1 - R_k / an) \quad (n = 1, 2, \ldots) \tag{2.3}$$

Because of the linearity of the problem it is sufficient to examine collision of the spheres and motion in one direction separately. If in conformity with this fact we consider $u_1 = \pm u_2$ and introduce the variable $\xi = \tan 1/2\, \theta$, from (2.1)-(2.3) and (1.7), (1.8) it is possible to obtain an expression for the tangential velocity v_θ on the sphere

$$v_\theta = \frac{1}{2} u_i \sin\theta_i + \frac{3}{2} u_i \sin\theta_i (1 + \xi^2)^{3/2} \times$$
$$\times \sum_{n=1}^{\infty} \left\{ \frac{n\gamma + 1}{[1 + (n\gamma + 1)^2 \xi^2]^{3/2}} \pm \frac{n\gamma - 1}{[1 + (n\gamma - 1)^2 \xi^2]^{3/2}} \right\}, \quad \gamma = \frac{2(R_1 + R_2)}{R_k} \tag{2.4}$$

The plus sign in (2.4) corresponds to head-on motion at the same speeds, and the minus sign corresponds to motion in one direction.

If we choose the plus sign, the asymptotics of the series in (2.5) can be easily be found as $\xi \to 0$ ($\theta \to 0$) by means of the Euler-Maclaurin formula, according to which we get

$$v_\theta = 2u_i R_k / (R_1 + R_2)\theta_i + O(\mathrm{const})$$

Consequently, as a result of approach of the colliding spheres to contact, a plane source was formed at the point of tangency of the spheres which throws back the fluid in the plane of tangency.

When the velocities of the contacting spheres are directed in the same senses $u_1 = -u_2$, it is convenient to change over to a coordinate system moving together with the spheres and to rewrite (2.4) in the following form: /662

$$v_\theta = -{}^3/_2 \, (1 + \xi^2)^{3/2} \, f(\xi) \, u_i \sin\theta_i \qquad (\xi = \mathrm{tg}\, {}^1/_2\theta) \tag{2.5}$$

$$f(\xi) = \sum_{n=-\infty}^{\infty} g(\xi, n - 1/\gamma), \qquad g(\xi, x) = \gamma x [1 + (x\gamma\xi)^2]^{-3/2} \tag{2.6}$$

It is proved below that as $\xi \to 0$

$$f(\xi) = -\frac{2\pi}{3}(4\pi + {}^3/_4\gamma\xi)\left(\sin\frac{2\pi}{\gamma}\right)\exp\left(-\frac{2\pi}{\gamma\xi}\right) / \gamma^{5/2}\xi^{7/2} \tag{2.7}$$

From formulas (2.5) and (2.7) it follows that the tangential velocity on the sphere falls off exponentially with decreasing distance to the point of contact. Thus, in the case of spheres of the same radius, the velocity near the contact zone varies as $\exp(-\pi/\theta)/\theta^{5/2}$. Consequently, the fluid stagnates near the zone of sphere contact.

Asymptotic (2.7) is obtained from (2.6) by means of the Poisson summation formula, which has the form [6]

$$f(\xi)=\sum_{n=-\infty}^{\infty} g(\xi, n-1/\gamma)=\sum_{l=-\infty}^{\infty} e^{-2\pi i l/\gamma} \int_{-\infty}^{\infty} e^{-2\pi i l x} g(\xi, x)\, dx \tag{2.8}$$

The integrals in (2.8) will be denoted by I_l; $I_l = -I_{-l}$, since $g(\xi, x) = -g(\xi, -x)$. The exchange of variable $x\xi\gamma = t$ is made in the integrals. Then

$$I_l=\frac{1}{\gamma\xi^2}\int_{-\infty}^{\infty} t(1+t^2)^{-5/2} e^{i\sigma t}\, dt \quad \left(\sigma=-\frac{2\pi l}{\xi\gamma}\right) \tag{2.9}$$

Before going over to a new method of integration, it is necessary to integrate (2.9) by parts, and then rectify the integrand at the point $t = i$ if $\sigma > 0$, in order to make the integral over the imaginary axis convergent up to the point $t = i$. As a result, I_l takes the form

$$I_l=\frac{i\sigma}{3\gamma\xi^2}\int_{-\infty}^{\infty}\left[\frac{1+it}{(1+t^2)^{3/2}}+\frac{\sigma}{(1+t^2)^{1/2}}\right] e^{i\sigma t}\, dt \tag{2.10}$$

Chosen for transformation (2.10) is a contour consisting of the following parts: from $-R$ to $+R$ along $\operatorname{Im} t = 0$; a part of the circle $Re^{i\theta}$, $\theta\in[0, 1/2\pi]$; the segment from $iR + \varepsilon$ to $i + \varepsilon$; the part of the circle $\varepsilon e^{i\theta} + i$, $\theta\in[-\pi, 0]$; the segment from $i - \varepsilon$ to $iR-\varepsilon$; a part of the circle $Re^{i\theta}$, $\theta\in[1/2\pi, \pi]$; (R and ε are real numbers). The function $(1 + t^2)^{1/2}$ takes the value $-i(y^2 - 1)^{1/2}$ on the segment $iy - \varepsilon$ and $i(y^2 - 1)^{1/2}$ on the segment $iy + \varepsilon$. As $R \to \infty$, $\varepsilon \to 0$ it follows, on the basis of the Cauchy theorem, that

$$I_l=\frac{2i\sigma}{3\gamma\xi^2}\int_{1}^{\infty}\frac{1+\sigma(y+1)}{(1+y)(y^2-1)^{1/2}}\, e^{-\sigma y}\, dy$$

The exchange of variable $y = 1 + u^2$ and the series expansion of the resultant integrand function make it possible to calculate the first terms of the asymptotic I_l as $\sigma \to \infty$ ($\xi \to 0$):

$$I_l=\tfrac{1}{3}i\pi\sqrt{-l}\,(\tfrac{3}{4}\xi\gamma-4\pi l)\exp\left(\frac{2\pi l}{\xi\gamma}\right)\gamma^{-3/2}\xi^{-5/2} \tag{2.11}$$

Here allowance has been made for the fact that $\sigma = -2\pi l/\xi\gamma$. The formula (2.11) is valid only for $l < 0$. Making use of the fact that I_l is an odd function, we can obtain formula (2.7) from (2.11) and (2.8), taking only $l = \pm 1$ into account.

3. The Kinetic Energy of the Fluid. The kinetic energy T of an ideal incompressible fluid is expressed, as is well known [3, 4], in terms of the values of the potential on the boundary surfaces /663

$$\frac{2}{\rho} T = \int_S \Phi \frac{\partial \Phi}{\partial n} ds \tag{3.1}$$

The motion of two spheres can always be represented as the sum of three motions: motion of the spheres along a line joining centers and motion along two mutually orthogonal directions which are orthogonal to the line joining centers. The kinetic energy of a fluid for arbitrary motion of the spheres is equal to the sum of the kinetic energies of the fluid in each of these three motions separately [1, 2]. This property of additivity can be proved by means of symmetry considerations [1, 2] or on the basis of the simplest properties of the potential and of Green identities. The additivity of the kinetic energy makes it possible to limit oneself to a calculation of the kinetic energy for two cases: motion of spheres along a line joining centers and perpendicular to the line at coplanar velocities.

When the spheres move only along the center line, substitution of formulas (1.2) and (2.1) into (3.1) after calculation of the integral

$$\int_0^{\pi} \frac{(R_i^2 - a_{in}^2) \cos\theta \sin\theta}{a_{in}(R_i^2 - 2R_i a_{in} \cos\theta + a_{in}^2)^{3/2}} d\theta = \frac{2}{R_i^2}$$

permits us to find the kinetic energy of the fluid:

$$\frac{1}{2\pi\rho} T = -\sum_{i=1}^{2} u_i \left(\frac{1}{3} \alpha_0^i + \sum_{n=1}^{\infty} \alpha_n^i \right) \tag{3.2}$$

This problem was solved by Hicks in a somewhat different form [1, 2]. Here the $\alpha_n^{c'}$ are known from (1.7) and (1.9) as functions of τ. The relationship between τ and a is given by formula (1.10)

The kinetic energy is a quadratic form in the velocities

$$\frac{1}{\pi\rho} T = A_1 u_1^2 + 2Bu_1u_2 + A_2u_2^2 \tag{3.3}$$

The coefficients A_i, B can be written in accordance with (1.7) and (3.3) in the form

$$\frac{A_i}{R_i^3} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{(A_n^i)^3}, \qquad \frac{B}{R_k^3} = \sum_{n=1}^{\infty} \frac{1}{(B_n^i)^3} \tag{3.4}$$

where A_n^i and B_n^i are known from (1.9).

When the spheres are in contact, the coefficients A_i, B take on a particularly simple form if (2.2) is taken into account:

$$\frac{A_i}{R_i^3} = \frac{1}{3} + \sum_{n=1}^{\infty} \left(1 + n\frac{R_1 + R_2}{R_k}\right)^{-3}, \quad B = \zeta(3)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-3}$$

where $\zeta(x)$ is Riemann's zeta function. In particular, if the radii of the spheres are equal, it is not difficult to calculate

$$A = R^3\,(^7/_8\zeta(3) - {}^2/_3) \approx 0.385\,R^3, \quad B = 0.125\zeta(3)R^3 \approx 0.150\,R^3$$

This coincides with an analogous result in [1].

4. The Forces of Hydrodynamics Interaction between Two Spheres at Short Distances. The motion of spheres in an ideal incompressible fluid is described by Lagrange equations [3, 4]; therefore, $\partial T/\partial a$ will be the force of interaction between two spheres. Hicks [1] found that the series determining the coefficients of quadratic form $\partial T/\partial a$ deverge if one sphere touches the other. The two leading terms of the asymptotics of sums of series can be obtained as follows. Denoting the general term of one of the series for dA_i/da or dB/da, which were determined by term-by-term differentiation of formula (3.4), by $f(n, \tau)$, it can be seen that the functions /664

$$\sum_{n=E[1/(\tau-1)]}^{\infty} f(n, \tau), \qquad \sum_{n=1}^{E[1/(\tau-1)]} [f(n, \tau) - f(n, 1)]$$

where $E(1/(\tau - 1))$ is the integral part of $(\tau - 1)^{-1}$, are bounded as $\tau \to 1$. Moreover, the difference $f(n, \tau) - f(n, 1)$ tends to zero uniformly with respect to n as $\tau \to 1$ owing to the fact that $f(n, \tau)$ tends to zero uniformly with respect to τ as $n \to \infty$. This is fulfilled despite the fact that the function $f(n, \tau)$ is not a uniformly continuous function of the argument τ. Because of the remarks we have made, the last two series in the indentity

$$\sum_{n=1}^{\infty} f(n, \tau) = \sum_{n=1}^{E[1/(\tau-1)]} f(n, 1) + \sum_{n=1}^{E[1/(\tau-1)]} [f(n, \tau) - f(n, 1)] + \sum_{n=E[1/(\tau-1)]}^{\infty} f(n, \tau)$$

can be replaced by integrals. Consequently, the following formula holds:

$$\sum_{n=1}^{\infty} f(n, \tau) = \sum_{n=1}^{E[1/(\tau-1)]} f(n, 1) - \int_1^{1/(\tau-1)} f(x, 1)\,dx + \int_1^{\infty} f(x, \tau)\,dx + O(\tau - 1) \tag{4.1}$$

When applying this formula to series (3.4), which determines the coefficients ∂T/ /∂a, the series on the right-hand side of (4.1) can be divided rather simply into a divergent part and a constant. The integrals in (4.1) are calculated, and quantities of two higher orders in (τ - 1) remain in the resultant expressions. As a result of the calculations we get, after several cumbersome computations,

$$p = R_1R_2 / (R_1 + R_2), \quad d = {}^2/_3 - {}^1/_2 \ln 2 - c, \quad \delta = a - R_1 - R_2$$

$$\frac{1}{p^2}\frac{dA_i}{da} = d + \frac{1}{2}\ln\frac{\delta}{p} - \sum_{n=1}^{\infty}\left[\frac{n(n+1)(n-1+3p/R_i)}{(n+p/R_i)^4} - \frac{1}{n}\right] \tag{4.2}$$

$$\frac{1}{p^2}\frac{dB}{da} = d + \frac{1}{2}\ln\frac{\delta}{p} + (1 - 3p^2/R_1R_2)\,\zeta(3)$$

where c is the Euler constant. In the case of spheres of same radius moving head-on at the same speed, we have at short distances, according to (4.2), (3.3), (3.4),

$$\partial T / \partial a \approx [{}^1/_2 \ln (a / R - 2) - 0.0948]\, \pi\rho u^2R^2 \tag{4.3}$$

From formulas (4.2) it is evident that the difference dA_i/da-dB/da remains finite /665 as a → $R_1 + R_2$, when the spheres approach to contact. It can be shown that the difference is always positive, i.e., spheres moving in one direction move away from each other at any ratio of radii. From (3.3), (3.4), (4.2) it follows that spheres of equal radii moving in contact in one direction are pushed apart by forces

$$\partial T / \partial a = ({}^3/_4\zeta(3) - \ln 2)\, \pi\rho u^2R^2 \approx 0.2084\pi\rho u^2R^2 \tag{4.4}$$

5. Velocity Potential upon the Motion of Spheres Perpendicular to a Line Joining Centers. When spheres move perpendicular to a line joining centers ($u_1 = u_2 = 0$), the zero approximation for the potential that satisfies (1.2) has the following form in the i-th coordinate system:

$$\Phi_0^i = -\frac{R_i^3}{2r_i^2} v_i \sin\theta_i \sin\varphi_i, \quad \Phi_0^k = -R_k^3 v_k \frac{r_i \sin\theta_i \sin\varphi_i}{2(r_i^2 - 2ar_i\cos\theta_i + a^2)^{3/2}} \tag{5.1}$$

It is well known [1, 2] that the potential is determined by a certain system of dipoles located within the spheres along the center line and orthogonal to this line. The problem consists in finding this entire system. For constructing the solution it is convenient to introduce dipole coordinates dependent on the dimensionless variables x_n. The dipole coordinate $b_{in} = b_{in}(a, x_1, \ldots, x_n)$ is determined analogously to (1.5):

$$b_{in} = R_i^2 x_n (a - b_{kn-1})^{-1}, \quad b_{i0} = 0, \quad x_n \in [0,1] \quad (n = 1, 2, \ldots) \tag{5.2}$$

It is easy to see that $b_{in} \le a_{in}$ always; $b_{in} = a_{in}$ only if $x_1 = 1, \ldots, x_n = 1$. The dipole located in the i-th sphere and having the coordinate b_{in} is written in the form

$$Q_n^i = \dot{r}_i \sin\theta_i \sin\varphi_i (r_i^2 - 2r_i b_{in} \cos\theta_i + b_{in}^2)^{-3/2} \tag{5.3}$$

It can be shown that for fixed n the functions

$$\Phi_{n-1}^k = Q_{n-1}^{k}, \quad \Phi_n^i = \left(\frac{b_{in}}{R_i}\right)^3 \left(Q_n^i - \int_0^1 Q_n^i x_n \, dx_n\right) \tag{5.4}$$

satisfy Eq. (1.3).

Here $x_n = 1$ in the functions outside the sign of the integral. It is easy to see that any Φ_n^i entering into (1.3) can be constructed by n-fold application of formulas (5.4) to the functions of the zeroth approximation (5.1). But in order to write the expression for the function Φ_n^i, it is necessary to introduce the coefficients $\beta_n^i = \beta_n^i(a, x_1, \ldots, x_{n-1})$, which are a generalization of the coefficients α_n^i that arise in solving the problem of the motion of spheres along a line joining centers*

$$\begin{aligned} 2\beta_{2n}^i &= v_i R_i^{3-3n} R_k^{-3n} (a_{k1} b_{i2} b_{k3} \ldots b_{i2n})^3 \\ 2\beta_{2n-1}^i &= v_k R_i^{-3n} R_k^{6-3n} (a_{i1} b_{k2} b_{i3} \ldots b_{i2n-1})^3 \end{aligned} \tag{5.5}$$

It can be observed that $|\beta_m^i| \leq |\alpha_n^i|$, the equality sign being reached when $x_1 = 1$, $\ldots$, $x_{n-1} = 1$. In addition to the new coefficients, it is necessary to introduce the oper- /666 ator L_n, defined as follows:

$$L_n f(x_n) = f(1) - \int_0^1 f(x_n) x_n \, dx_n \quad (n = 1, 2, \ldots) \tag{5.6}$$

Now, by means of (5.1) and (5.4)-(5.6), it is possible to write Φ_n^i in the compact analytic form

$$\Phi_0^i = -{}^1/_2 v_i R_i^3 Q_0^i, \quad \Phi_n^i = -L_1 \ldots L_{n-1} \beta_n^i L_n Q_n^i \tag{5.7}$$

By this very fact the problem of finding the velocity potential is solved.

In order to calculate the kinetic energy of the fluid it is sufficient to know the potential Φ on the spheres. The expression for the potential on a sphere is simplified

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*Here, the argument $x_m = 1$ is in each b_{im} $(m = 1, 2, \ldots)$.

if account is taken of the fact that

$$Q^k_{n-1} = (b_{in} / R_i)^3 Q^i_n \text{ when } r_i = R_i$$

and, consequently, $\Phi^k_{n-1} = -L_1 \ldots L_{n-1}\beta^{i'}_n Q^i_n$. Then, according to (1.3) and (5.7), the potential on a sphere is

$$\Phi|_{R_i} = -\frac{1}{2} v_i R_i^3 Q^i_0|_{R_i} - \sum_{n=1}^{\infty} L_1 \ldots L_{n-1}\beta^i_n \left(2Q^i_n - \int_0^1 Q^i_n x_n \, dx_n\right)\Big|_{R_i} \tag{5.8}$$

6. Kinetic Energy of the Fluid upon Motion of the Spheres Perpendicular to the Center Line. When the spheres move perpendicular to the center line at coplanar speeds ($u_1 = u_2 = 0$), the kinetic energy is calculated from formulas (1.2), (3.1), (5.3) and (5.8). In this case it is sufficient to consider that

$$\int_{S_i} \left(2Q^i_n - \int_0^1 Q^i_n x_n \, dx_n\right) \sin\theta_i \sin\varphi_i \, ds = 2\pi$$

and the kinetic energy is written in the following form:

$$\frac{1}{\pi\rho} T = \sum_{i=1}^{2} \left(\frac{1}{3} v_i^2 R_i^3 + \sum_{n=1}^{\infty} L_1 \ldots L_{n-1}\beta^i_n v_i\right) \tag{6.1}$$

where the operator L_n is defined by formulas (5.6), the coefficient β^i_n is known from (5.5), the operator L_0 is equal to the unit operator.

It can be proved that the series on the right side of (6.1) converges faster than the analogous series on the right side of (3.2), which converges approximately as $1/n^3$. To do so we first show that the continued fraction b_{in} defined by (5.2), decomposes into a convergent (n-1)-dimensional power series in $x_1, \ldots, x_{n-1}$ with nonnegative coefficients. This is not difficult to see if the continued fractions $b_{i1}, b_{i2}, \ldots,$ are decomposed into a series, one after the other, using formula (5.2). From the fact that all b_{in} decompose into convergent power series with nonnegative coefficients in the (n-1)-dimensional cube /667 $x_1 \in [0, 1], \ldots, x_{n-1} \in [0, 1]$, it follows that $|\beta^i_n|$, defined by formula (5.5), also decomposes into a convergent power series with nonnegative coefficients in the same cube:

$$|\beta^i_n| = \sum_{m_1, \ldots, m_{n-1}} C^i_{m_1, \ldots, m_{n-1}} x_1^{m_1} \ldots x_{n-1}^{m_{n-1}}, \quad C^i_{m_1, \ldots, m_{n-1}} \geqslant 0 \tag{6.2}$$

Relatively simple calculations on the basis of (5.6) and (5.2) yield

$$0 < |L_1 \ldots L_{n-1}\beta_n^i| = \sum_{m_1,\ldots,m_{n-1}} C^i_{m_1,\ldots,m_{n-1}} \frac{(m_1+1)\ldots(m_{n-1}+1)}{(m_1+2)\ldots(m_{n-1}+2)} <$$

$$< \sum_{m_1,\ldots,m_{n-1}} C^i_{m_1,\ldots,m_{n-1}} = |\alpha_n^i|$$

where α_n^i is determined by formula (1.6) for $u_i = v_i$. From the inequalities have obtained it is evident that for any distances between spheres the kinetic-energy series (6.1) is majorized by kinetic-energy series (3.2) if we set $u_i = v_i$. Thus, the kinetic-energy series upon motion of spheres perpendicular to a center line converges faster than the kinetic-energy series upon motion of spheres along a line joining centers.

Formulas (5.5), (5.6), (6.1) make it possible to find the coefficients A_i' and B' of the kinetic energy $T = A_1'v_1^2 + 2B'v_1v_2 + A_2'v_2^2$. Thus, for spheres of equal radius in the case of contact we get, according to these formulas,

$$A' = 0.347\pi\rho R^3, \qquad B' = 0.067\pi\rho R^3$$

The kinetic energy of a fluid upon the motion of two identical spheres in contact at the same speed perpendicular to a line joining centers proves to be $T = 0.828\pi\rho u^2 R^3$.

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